ANALYSIS AND COMPARISONS OF SOME SOLUTION CONCEPTS FOR STOCHASTIC PROGRAMMING PROBLEMS

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ABSTRACT**:** The aim of this study is to analyse the resolution of Stochastic Programming Problems in which the objective function depends on parameters which are continuous random variables with a known distribution probability. In the literature on these questions different solution concepts have been defined for problems of these characteristics. These concepts are obtained by applying a transformation criterion to the stochastic objective which contains a statistical feature of the objective, implying that for the same stochastic problem there are different optimal solutions available which, in principle, are not comparable. Our study analyses and establishes some relations between these solution concepts.

KEY WORDS**:** Stochastic Programming. Optimal solution concepts.

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1. Introduction

When a real problem is modelled and solved by means of a mathematical programming problem it may happen that some of the parameters which figure in the problem are unknown, whether it be in the objective function or in the feasible set. If these parameters of unknown value can be taken as random variables the resulting problem is a stochastic programming one. Obviously, the fact that one of the constraint functions or the objective function is affected by random parameters gives rise to the view that that function is also a random variable and, given that, in general, a random variable does not admit a relation of order, it is necessary to specify a solution concept for these problems.

The characteristics of stochastic programming problems and the determining of optimal solutions for them have been widely studied in the literature. Among other studies we should make special mention of the books of Prékopa (1995), Kall and Wallace (1994), Kibzun and Kan (1996) or the articles of Kall (1982), Leclerq (1982) and Zare and Daneshmand (1995). In these studies we can observe how the existing solution concepts for stochastic programming problems originate from the transformation of the problem into a deterministic one which is given the term *deterministic equivalent*. This transformation is achieved by taking statistical features of the objective function or of the constraint functions dependent on random parameters in the problem.

In this study we focus our attention on problems of stochastic programming in which the random parameters affect only the objective function of it or in which, if they affect the constraints, this latter has already been transformed into its deterministic equivalent by one of the existing procedures in stochastic programming. In addition, we assume that they are continuous random variables, so that the stochastic programming problem which we are considering is as follows:

$$\underset{\mathbf{x} \in D}{\text{Min}}\ \tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \tag{1.1}$$

where $\mathbf{x}$ is the vector of decision variables, $D \subset \mathbb{R}^n$ is a closed set, bounded, convex and not empty, $\tilde{\mathbf{c}}$ is a random vector defined on the set $E \subset \mathbb{R}^s$, with known probability distribution and whose components are continuous random variables. Furthermore, we assume that the probability distribution of vector $\tilde{\mathbf{c}}$ is independent of the decision variables of the problem, $x_1, \ldots, x_n$.

From the hypotheses established for the problem (1.1) it follows that the objective function of the problem depends on random parameters and, as previously indicated, this implies that the function is a random variable.

In order to solve the problem (1.1), different solution concepts have been defined in the literature. Each of these concepts proceeds from the application of a criterion to transform the stochastic objective into a deterministic function. These criteria draw on some statistical feature from the stochastic objective, so that the problem (1.1) is attributed various deterministic equivalents and, in general, various possible optimal solutions. The diversity of concepts and solutions can give rise in some way to certain confusion when a problem is being solved and immediately poses various questions, such as the existing differences between some concepts and others, whether any of the concepts so far defined is better than another, etc. In this study we consider five solution concepts to the problem (1.1) which correspond to the application of different criteria to transform the stochastic problem into its deterministic equivalent, all of which have been defined previously, and some relations are established between them. We begin by defining in Section 2 these five solution concepts and establishing some differences between them. In Section 3 some relations between these concepts are established and in Section 4 we pose and solve an example to illustrate the results obtained.

2. Some Solution Concepts for Stochastic Programming Problems

Let us consider the stochastic programming problem (1.1). Let $\bar{z}(\mathbf{x})$ be the expected value function of $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$ and $s^2(\mathbf{x})$ its variance, which we assume to be finite for all $\mathbf{x} \in D$. Let us now look at five optimum concepts for this problem. As previously indicated, each one of these concepts derives from the application of transformation criteria to the objective function of the problem and the obtaining of a deterministic equivalent problem from the initial problem. The transformation criteria which we consider are: expected value, minimum variance, expected value standard deviation efficiency, minimum risk and Kataoka.

Definition 1: Expected value solution

Point $\mathbf{x} \in D$ is the expected value solution to (1.1) if it is the optimal solution to the problem:

$$\underset{\mathbf{x}\in D}{\text{Min}} \quad \bar{z}(\mathbf{x}) \tag{2.1}$$

In other words, the expected value optimal solution to the problem (1.1) is simply the optimal solution resulting from the "substitution" of the stochastic objective function by its expected value, thus taking up this measurement of central tendency of the random variable for its optimisation. This solution concept has been frequently used in the literature for the resolution of stochastic programming problems, although this criterion cannot always be considered "appropriate" given that if we remain with only the expected value of the random variable, certain features of the stochastic objective function of the problem might not be included in the deterministic equivalent problem. In this sense, some authors such as Kataoka (1963) or Prékopa (1995) make some critical comments about the application of the above criterion and establish same conditions under which it can be considered “appropriate”.

Definition 2: Minimum variance solution

Point $\mathbf{x} \in D$ is the minimum variance solution to the problem (1.1) if it is the optimum of the problem:

$$\underset{\mathbf{x}\in D}{\text{Min}} \quad s^{2}(\mathbf{x}) \tag{2.2}$$

In other words, in order to solve the stochastic programming problem, the criterion considered is that of minimising the variance of the stochastic objective function $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$, choosing, therefore, the vector $\mathbf{x}$ for which the random variable $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$ is more concentrated around its expected value. This implies that the optimisation criterion is that of the minimum, regardless of whether the problem of optimisation be that of the minimum (a hypothesis which we sustain in this study) or of the maximum, and in some ways, can be considered a criterion of “low” risk.

Definition 3: Expected value standard deviation efficient solution

Point $\mathbf{x} \in D$ is a expected value standard deviation efficient solution to the problem (1.1) if it is an efficient solution in the Pareto sense to the following bi-criteria problem:

$$\underset{\mathbf{x}\in D}{\text{Min}} \ \left(\bar{z}(\mathbf{x}),\ s(\mathbf{x})\right) \tag{2.3}$$

which includes the expected value and the standard deviation of the stochastic objective function. We denote E_{Es} as the set of efficient solutions to the problem (2.3) and E_{Es}^{p} as the set of properly efficient solutions to this problem.

This concept of efficiency was introduced by Markowitz (1952) as a means of solving problems in the field of portfolio selection in finance economics. Markowitz considers the bi-criteria problem of expected value minimum variance, that is:

$$\underset{\mathbf{x}\in D}{\text{Min}} \left(\overline{z}(\mathbf{x}),\ s^2(\mathbf{x})\right) \tag{2.4}$$

In this study we have chosen to substitute the variance of the stochastic objective for its standard deviation given that, in this way, the two objectives of the problem (1.1) are expressed in the same units of measurement. In any case, since the square root function is strictly increasing, the efficient sets of both problems coincide.

Finally, we define the concepts of optimal solution minimum risk of aspiration level u and the optimal solution with probability β. Both solutions are obtained by applying the minimum risk and Kataoka criteria, respectively, referred to in the literature as criteria of maximum probability or satisfying criteria, due to the fact that, as we shall see in the following definition, in both cases the criteria to be used provide, in one way or another, "good" solutions in terms of probability.

Definition 4: Minimum risk solution of aspiration level u

Point $\mathbf{x} \in D$ is the minimum risk solution of aspiration level u for the problem (1.1) if it is the optimal solution to the problem:

$$\underset{\mathbf{x}\in D}{\text{Max}}\ P\left(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u\right) \tag{2.5}$$

That is, to obtain a minimum risk optimal solution to the problem (1.1) we apply what in the literature is referred to as the minimum risk criterion. This consists of fixing a level for the stochastic objective function $u \in \mathbb{R}$, to which we apply the term *aspiration level*, and of maximising the probability that the objective will be less than or equal to that level: $P\left(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u\right)$. In this way, the fixed level u can be interpreted as being the uppermost level that the Decision Maker (DM) is capable of admitting for the stochastic objective that we wish to minimise.

The first studies in which this criterion, also termed model P, is proposed as a means of solving the problem (1.1), are those carried out by Charnes and Cooper (1963) and Bereanu (1964).

Definition 5: Kataoka solution with probability b

Point $\mathbf{x} \in D$ is the Kataoka solution with probability b to the problem (1.1) if there exists $u \in \mathbb{R}$ such that $(\mathbf{x}^{\mathbf{t}}, u)^{t}$ is the optimal solution to the problem:

$$\begin{aligned} &\operatorname*{Min}_{\mathbf{x},u} \; u \\ \text{s.t} \;\; &P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u) = \boldsymbol{b} \\ &\mathbf{x} \in D \end{aligned} \tag{2.6}$$

Thus, the obtaining of optimal solutions with probability b is derived from the application of the b-fractile or Kataoka criterion to the problem (1.1), introduced by Kataoka (1963). It consists of fixing a probability b, for the objective and determining the lowest level u, which cannot exceed the objective function with that probability. The objective function of the stochastic problem goes on to become part of the feasible set as a probabilistic or chance constraint and the deterministic equivalent problem is a problem with n+1 decision variables: the n decision variables of the stochastic programming problem carried in vector $\mathbf{x}$ and variable u.

Once these five solution concepts for stochastic programming problems have been defined, each one associated with a criterion which is considered “adequate” for the resolution of the initial problem, a comparative analysis of the above-mentioned concepts enables us to comment on the following differences which emerge between them:

1. In order to apply the maximum probability criteria (minimum risk and Kataoka) it is necessary to fix previously a parameter: level u, if we apply the minimum risk criterion, or probability b, if we apply the Kataoka criterion,

while in the first three criteria this is unnecessary. Moreover, the minimum risk and Kataoka optimal solutions will depend, generally, on the values assigned to those parameters.

2. Whilst the first three solution concepts (expected value, minimum variance and expected value standard deviation efficiency) can be obtained provided the expected value and variance of the objective function are known, the two criteria of maximum probability depend on the distribution function of the stochastic objective function and, therefore, in order to obtain them it is essential to know the latter.
3. The last three solution concepts (expected value standard deviation efficiency and the two related to maximum probability) give rise, in general, to a set of solutions: the set of expected value standard deviation efficient solutions for the first of these and a solution for each level, u, or probability β to be fixed for the last two concepts which are not generally comparable.

From what has been stated above it can be concluded that the diversity of solution concepts leads to the need to choose one or the other to solve the stochastic programming problems and, in this sense, we can state that the resolution of these problems always implicitly contains a decision process. Obviously, the choice of a solution will have to be made bearing in mind the characteristics of the problem to be solved and the preferences of the DM. However, as we shall see below, if specific hypotheses are shown to hold true, the solutions which we have just defined are related, although at the beginning one may think the opposite, due to the fact that these are solutions to deterministic equivalent problems which draw on statistical characteristics different from the stochastic objective.

3. Relations between the Solution Criteria for Stochastic Programming Problems

In this section we deal with the analysis of the existence of relations between the solutions to the problem (1.1) previously defined, which are obtained by applying to the problem the criteria discussed in the previous section: expected value, minimum variance, expected value standard deviation efficiency, minimum risk and Kataoka. Specifically, relations are established between the minimum risk optimal solutions and the Kataoka ones and between these and the expected value standard deviation efficiency solutions for a stochastic programming problem. Prior to that, we shall consider some of the results of Multiple Objective Programming which will be applied throughout this section. In addition, as an immediate consequence of these results we shall establish relations between the optimal solutions expected value and minimum variance and the expected value standard deviation efficient solutions.

Let us consider the following multiple objective programming problem:

$$\underset{\mathbf{x} \in D}{\text{Min}} \left(f_1(\mathbf{x}), \ldots, f_q(\mathbf{x}) \right) \tag{3.1}$$

where **f** is a vectorial function, $\mathbf{f}: H \subset \mathbb{R}^n \rightarrow \mathbb{R}^q$, and the problem resulting from the application of the weighting method to it gives us:

$$\underset{\mathbf{x} \in D}{\text{Min}} \; m_1 f_1(\mathbf{x}) + \ldots + m_q f_q(\mathbf{x}) \tag{3.2}$$

with vector of weights $\mathbf{m}$, $m_k \geq 0$, for all $k \in \{1, 2, \ldots, q\}$, which, without losing generality, we assume to be normalised.

The following theorem relates to the efficient solutions to the multiple objective problem (3.1) and the optimal solutions to the associated weighted problem, (3.2).

Theorem 1.(Sawaragi, Nakayama and Tanino (1985).

Let us assume that the functions $f_1,..., f_q$ are convex and that D is a convex set. Thus:

a) If **x*** is a properly efficient solution for the multiple objective problem (3.1), there exists a weight vector $\mathbf{m}$ with strictly positive components such that **x*** is the optimal solution for the weighted problem (3.2).

b) If **x*** is the optimal solution to the weighted problem (3.2), for a vector of weights with strictly positive components, **x*** is a properly efficient solution for the multiple objective problem (3.1).

c) If **x*** is the only solution to the problem (3.2), with $\mathbf{m} \geq \mathbf{0}$, then **x*** is an efficient solution to the problem (3.1). If it is not the only solution, the solutions obtained are weakly efficient for (3.1).

This theorem of multiple objective programming allows us to relate the expected value and minimum variance solutions to the problem (1.1) to the expected value standard deviation efficient set. Therefore, let us consider the problems (2.1), (2.2) and (2.3). The weighted problem associated to the problem (2.3), for weights m and 1- m , $m \in$ [0, 1], is:

$$\underset{\mathbf{x}\in D}{\text{Min}}\; m\,\bar{z}(\mathbf{x}) + (1-m)\,s(\mathbf{x})$$

From Theorem 1 we can state that:

1. If the expected value optimal solution to the problem (1.1) is unique then it is an expected value standard deviation efficient solution. Where it is not unique it can only be assured that the expected value optimal solutions are expected value standard deviation weakly efficient solutions, but we cannot state that they are expected value standard deviation efficient solutions.
2. If the variance of the stochastic objective is a strictly convex function, the minimum variance problem has a unique solution (since we assume that set D

is closed, bounded, convex and not empty), and so its optimal solution is an expected value standard deviation efficient solution to the stochastic programming problem. Should the minimum variance problem have more than one optimal solution, these solutions are expected value standard deviation weakly efficient solutions and the only thing we can be certain (as in point 1 above) is the weak efficiency of these solutions.

We shall now analyse the relations between the minimum risk and Kataoka problems. Following this, we shall return to deal with the concept expected value standard deviation efficient solution and we shall relate it to the Kataoka and minimum risk solutions to the stochastic programming problem.

3.1. Relations between the minimum risk problem and the Kataoka problem

Given the stochastic programming problem (1.1) let us consider the deterministic equivalent problem (2.5) corresponding to the criterion of minimum risk of aspiration level u, and the problem (2.6) corresponding to the application of the Kataoka criterion for a probability $b \in (0, 1)$.

The following theorems establish a relation between the optimal solutions to these two problems.

Theorem 2

Let us assume that the distribution function of the random variable $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$ is strictly increasing. Then $\mathbf{x}^*$ is the minimum risk solution of aspiration level u^* if and only if $(\mathbf{x}^{*\mathbf{t}}, u^*)^{\mathbf{t}}$ is the Kataoka solution with probability b^*, with u^* and b^* so that:

$$P(\tilde{z}(\mathbf{x}^*, \tilde{\mathbf{c}}) \leq u^*) = b^* .$$

Proof: We demonstrate the theorem by process of *reductio ad absurdum.*

$\Rightarrow$) If $\mathbf{x}^*$ is the solution to the problem (2.5) it is true that:

$$P\left(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*\right) \leq P\left(\tilde{z}(\mathbf{x}^*, \tilde{\mathbf{c}}) \leq u^*\right) = \boldsymbol{b}^*, \ \forall \ \mathbf{x} \in D$$

Let us assume that $(\mathbf{x}^{*t}, u^*)^t$ is not the solution to the problem (2.6). In this case there exists a vector $(\mathbf{x}^t, u)^t$, feasible in (2.6) proving that $u < u^*$, that is:

$$\mathbf{x} \in D, \text{ there exist } u \in \mathbb{R} \text{ such that } P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u) = \beta^* \text{ and } u < u^*.$$

However as the distribution function of the random variable $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$ is strictly increasing , it is also true that:

$$P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*) = P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u) + P(u < \tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*) = \beta^* + q$$

where $q = P(u < \tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*) > 0$, from which we deduce that there exists a vector $\mathbf{x} \in D$ for which $P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*) > \beta^*$, which contradicts the hypothesis.

$\Leftarrow$) By hypothesis $(\mathbf{x}^{*t}, u^*)^t$ is the optimal Kataoka solution of level β^*, therefore it proves that:

a) It is feasible in (2.6): $\mathbf{x}^* \in D$ and $P(\tilde{z}(\mathbf{x}^*, \tilde{\mathbf{c}}) \leq u^*) = \beta^*$

b) If $(\mathbf{x}^t, u)^t$ proves that $\mathbf{x} \in D$ and $P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u) = \beta^*$, then $u^* \leq u$.

Let us assume that $\mathbf{x}^*$ is not the optimal solution to the problem (2.5). Then, there exists a vector $\mathbf{x} \in D$ for which:

$$P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u^*) > \beta^*$$

As the distribution function of the random variable $\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}})$ is strictly increasing, it holds true that there exists a $u < u^*$ for which:

$$P(\tilde{z}(\mathbf{x}, \tilde{\mathbf{c}}) \leq u) = \beta^*$$

which contradicts the hypothesis b).■

From Theorem 2 we can affirm that if the distribution function of the stochastic objective is a strictly increasing function there exists a reciprocity between the minimum risk and the Kataoka solutions, such that, if we have the minimum risk solution (Kataoka solution) we can affirm that it is the Kataoka solution (minimum risk solution) , that is:

1. For each fixed aspiration level u, the minimum risk solution is also the Kataoka solution with a probability b equal to the maximum probability obtained in the minimum risk problem
2. For each fixed b the Kataoka solution is also the minimum risk solution, if, for this latter, we establish an aspiration level u equal to the optimal value of the Kataoka problem.

3.2. Relations between the optimal solutions of Kataoka and the expected value standard deviation efficient solutions

We shall now analyse the existence of some relation between those solutions to the stochastic programming problem (1.1) which correspond to the application of the Kataoka criterion and the set of expected value standard deviation efficient solutions to the same problem.

Up till now, the relations that we have established between the solution criteria for stochastic programming problems have been of a general nature, that is, applicable to any problem of stochastic programming. However, in this section the relations to establish are exclusively applicable to stochastic programming problems of specific characteristics, given that for the Kataoka criterion to be applied it is necessary to know the distribution function of the stochastic objective, something which is, on the whole, complicated, even when the probability distribution of vector $\tilde{\mathbf{c}}$ is known. This fact leads us to the theme of our study, assuming the objective function of the problem to be of the linear type, so that from this point onwards, the problem we are dealing with is:

$$\operatorname*{Min}_{\mathbf{x}\in D} \ \tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x} \tag{3.3}$$

Furthermore, we establish a hypothesis regarding the probability distribution of vector $\tilde{\mathbf{c}}$, and, specifically, we consider two cases which correspond to the hypothesis that vector $\tilde{\mathbf{c}}$ is normally distributed (normal case) and to the consideration that vector $\tilde{\mathbf{c}}$ depends linearly on a single random variable (simple randomization case). In both cases it is possible to determine the probability distribution function of the stochastic objective $\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ and, therefore, to raise and solve the problem (3.3) by applying the Kataoka criterion.

To carry out this analysis we begin by raising the problem of expected value standard deviation efficiency of the problem (3.3) and the problems corresponding to the application of the Kataoka criterion in the cases mentioned, and, following this, we analyse the relations between them.

Let $\bar{\mathbf{c}}$ be the expected value vector of $\tilde{\mathbf{c}}$ and $\mathbf{V}$ be its variance and co-variance matrix, which we assume is positive definite. Then the expression of the expected value of $\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ is $\bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ and its variance is $\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x}$, and, therefore, the set of expected value standard deviation efficient solutions for the problem (3.3) is that of the problem:

$$\underset{\mathbf{x}\in D}{\text{Min}} \quad \left(\bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}, \quad (\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}\right) \tag{3.4}$$

As we assume that the feasible set D is convex, by hypothesis from the origin, and the function $(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}$ is convex (see Stancu-Minasian (1984) pages 94-95), the bi-criteria problem expected value standard deviation is convex, and we can state, on the basis of Theorem 1, that the set of expected value standard deviation properly efficient solutions to the problem (3.4), E^{p}_{Es}, can be generated from the resolution of the weighted problem:

$$\underset{\mathbf{x}\in D}{\text{Min}}\ m\bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} + (1-m)(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2} \tag{3.5}$$

where $m \in (0,\ 1)$. Subsequently we go on to obtain the deterministic equivalent problems for (3.3) corresponding to the application of the Kataoka criterion, in the cases previously cited.

a) Normal linear case

Let us suppose that $\mathbf{0} \notin D$ and $\tilde{\mathbf{c}}$ is a random vector multinormal with expected value $\bar{\mathbf{c}}$ and positive definite matrix of variances and co-variances $\mathbf{V}$.

In this case the random variable $\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ is a normal variable with expected value $\bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ and variance $\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x}$, which implies that:

$$P(\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x} \le u) = P\left(\left(\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x} - \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}\right)/(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2} \le \left(u - \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}\right)/(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}\right) = \Phi\left(\left(u - \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x}\right)/(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}\right)$$

where Φ is the standardised normal distribution function. This implies that the feasible set of the problem (2.6) is, according to the established hypotheses:

$$\left\{(\mathbf{x}^{\mathbf{t}}, u)^{\mathbf{t}} \in D \times \mathbb{R} \;\; / u = \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} + \Phi^{-1}(b)(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}\right\}$$

and the Kataoka problem, for a probability b, is:

$$\begin{aligned} &\operatorname*{Min}_{\mathbf{x},u} u \\ \text{s.a } \; & u = \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} + \Phi^{-1}(b)(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2} \\ & \mathbf{x} \in D \end{aligned}$$

equivalent to:

$$\operatorname*{Min}_{\mathbf{x}\in D} \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} + \Phi^{-1}(b)(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2}$$

If $a = \Phi^{-1}(b)$, we can express the preceding problem as follows:

$$\operatorname*{Min}_{\mathbf{x}\in D} \; \bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} + a\,(\mathbf{x}^{\mathbf{t}}\mathbf{V}\mathbf{x})^{1/2} \tag{3.6}$$

b) Simple linear randomization case

Let us assume that vector $\tilde{\mathbf{c}}$ depends linearly on a single random variable: $\tilde{\mathbf{c}} = \mathbf{c}^{\mathbf{1}} + \tilde{t}\,\mathbf{c}^{\mathbf{2}}$, where $\tilde{t}$ is a continuous random variable of expected value $\bar{t}$, standard deviation $u > 0$, $u < \infty$, and distribution function F_t, strictly increasing. Moreover, let us suppose that $\mathbf{c}^{\mathbf{2t}}\mathbf{x} > 0$ for all $\mathbf{x} \in D$. With these hypotheses the expected value of $\tilde{\mathbf{c}}$ is $\bar{\mathbf{c}} = \mathbf{c}^{\mathbf{1}} + \bar{t}\,\mathbf{c}^{\mathbf{2}}$ and its matrix of variances and co-variances is $\mathbf{V} = u^2\mathbf{c}^{\mathbf{2}}\mathbf{c}^{\mathbf{2t}}$.

Proceeding from the previous hypotheses it follows that:

$$P(\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x} \le u) = P(\tilde{t}\,\mathbf{c}^{\mathbf{2t}}\mathbf{x} \le u - \mathbf{c}^{\mathbf{1t}}\mathbf{x}) = P\left(\tilde{t} \le \left(u - \mathbf{c}^{\mathbf{1t}}\mathbf{x}\right)/\mathbf{c}^{\mathbf{2t}}\mathbf{x}\right) = F_t\left(\left(u - \mathbf{c}^{\mathbf{1t}}\mathbf{x}\right)/\mathbf{c}^{\mathbf{2t}}\mathbf{x}\right)$$

so that the Kataoka problem is equivalent to:

$$\underset{\mathbf{x}\in D}{\text{Min}}\ \mathbf{c^{1t}x} + F_t^{-1}(b)\mathbf{c^{2t}x}$$

As in the normal case, making $a = \left(F_t^{-1}(b) - \bar{t}\right)/u$ we can express the preceding problem by means of the problem (3.6).

Therefore, in the two cases analysed, the problem corresponding to the Kataoka criterion is problem (3.6), where $a = \Phi^{-1}(b)$ in the normal case and $a = \left(F_t^{-1}(b) - \bar{t}\right)/u$ in the simple randomization case.

In order to establish the relations between the optimal solution of the Kataoka problem in these two cases and the set of expected value standard deviation properly efficient solutions we have only to relate the optimal solutions of the problems (3.5) and (3.6).

Proposition 1

If $a > 0$ and $m \in (0, 1)$, with $m = 1/(1+a)$, then problems (3.5) and (3.6) are equivalent.

Proof: Since $m \in (0, 1)$ the solution to problem (3.5) coincides with that of:

$$\underset{\mathbf{x}\in D}{\text{Min}}\ \mathbf{\bar{c}^t x} + (1-m)(\mathbf{x^t V x})^{1/2}/m$$

and simply doing $a = (1-m)/m > 0$ or, what is equivalent, $m = 1/(1+a)$ we obtain the equivalence between problems (3.5) and (3.6).

Thus, from the Proposition 1, we establish an equivalence between problem (3.6) and the weighted problem (3.5). If we denote $\mathbf{x}^*(a)$ the optimal solution to problem (3.6), we can be sure that:

$$\bigcup_{a>0} \mathbf{x}^*(a) = \mathrm{E}_{Es}^{p}$$

Let us see what interpretation this result has in the cases analysed. In both cases the parameter α multiplies the standard deviation of the stochastic objective and depends on the fixed probability for the application of the Kataoka criterion. Specifically:

1. In the normal case $a = \Phi^{-1}(b)$, and consequently this parameter is positive if and only if the fixed probability, b, is greater than 0.5.
2. In the simple randomization case $a = \left(F_t^{-1}(b) - \bar{t}\right)/s_t$ and therefore $a > 0$ if the probability to be fixed in the Kataoka problem $b > F_t(\bar{t})$, that is greater than the probability of the random variable $\tilde{t}$ being less than or equal to the expected value.

Consequently, in both cases, the fact that a is strictly positive can be interpreted as fixing a "high" probability.

From the results obtained we can state that:

- If the probability fixed in the Kataoka problem is "high" ($b > 0.5$ in the normal case and $b > F_t(\bar{t})$ in the simple randomization case), we can affirm that the optimal solution to this problem is a expected value standard deviation properly efficient solution of the stochastic problem (1.1).
- Given a expected value standard deviation properly efficient solution to the problem (1.1), in the cases analysed there exists a "high" probability, b, such that this solution is a Kataoka optimum.

Obviously, based on these results one has to ask the question of what happens when, in either of these two cases under study, the fixed probability is such that $a \leq 0$. Note that for $a = 0$ the problem that is obtained in both cases is the expected value problem, previously mentioned in the study, and it corresponds to the fixing of a probability $b = 0.5$ in the normal case and $b = F_t(\bar{t})$ in the simple randomization case.

What type of solution is obtained when the probability fixed is “low”? When this is so the optimal solution to the Kataoka problem does not have to be a expected value standard deviation properly efficient solution to the stochastic problem. In this sense it must be noted that the expected value standard deviation efficiency criterion is appropriate when the individual is risk averse. So, in the cases analysed, the hypothesis of risk aversion is translated into the fixing of a “high” probability. In addition to this, if we analyse the objective function of the Kataoka problem, we note that when the fixed probability is low, the standard deviation of the stochastic objective is multiplied by a negative factor. This tells us that, in some way, at least in the cases analysed, the minimum variance criterion may not be appropriate if the DM wants a risky solution to the stochastic problem.

On the other hand, it must be emphasised that, as for these problems, the distribution function of the stochastic objective is strictly increasing, this determines as true the reciprocity analysed in the previous section amongst the optimal Kataoka and minimum risk solutions, so that it can be stated that that the minimum risk optimal solutions maintain with the set of expected value standard deviation properly efficient solutions, the same relations that we have obtained between these latter and the optimal Kataoka solutions.

Finally, we establish a further connection between the problems (3.6) and (3.5) in Proposition 2. Specifically we relate the weights assigned to the expected value and the standard deviation in the weighted problem (3.5) with the value of parameter a in problem (3.6), whose optimal solution coincides with that of (3.5). Given that the weight assigned, m, in problem (3.5) can be considered as a measurement of the relative importance which is given to the expected value of the objective function as opposed to the standard deviation in the obtaining of expected value standard deviation efficient solutions, we can also pose the question of whether there exists any relation between the value of the parameter m and the value of the parameter a which will weight the standard deviation in problem (3.6).

Proposition 2

Let $\mathbf{x}(m), \mathbf{x}(m'), \mathbf{x}(a)$ and $\mathbf{x}(a')$ be optimal solutions to the problems (3.5) for μ, (3.5) for m', (3.6) for a and (3.6) for a', respectively with m, m' $\in$ (0, 1), $a = (1 - m)/m$ and $a' = (1 - m')/m'$. Thus, $m < m'$ if and only if $a' < a$.

Proof:

⇒) Since m <m', it is evident that: 1- m' < 1-m and $1/m' < 1/m$, consequently:

$$a' = (1 - m')/m' < (1 - m)/m' < (1 - m)/m = a$$

⇐) Obvious from (⇒).■

Given that $a = \Phi^{-1}(b)$ in the normal case and $a = \left(F_t^{-1}(b) - \bar{t}\right)/s_t$ in the simple randomization case, the result is that a weight, m, higher for the expected value in the weighted problem (3.5), corresponds to a lower probability in the Kataoka problem and, therefore, more risky in terms of probability.

To illustrate the results obtained in this section, we shall now propose and solve the following example.

4. Example

Let us consider the following problem of stochastic programming:

$$
\begin{aligned}
&\underset{\mathbf{x}}{\text{Min}} \quad \tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x} = \tilde{c}_1 x_1 + \tilde{c}_2 x_2 \\
&\text{s. t.} \quad 3x_1 + 2x_2 \geq 2 \\
&\qquad\quad 2x_1 - 3x_2 \leq 8 \\
&\qquad\quad x_1 + x_2 \leq 5 \\
&\qquad\quad x_1, x_2 \geq 0
\end{aligned}
\tag{4.1}
$$

where $\tilde{\mathbf{c}} = \mathbf{c^1} + \tilde{t}\,\mathbf{c^2}$, with $\mathbf{c^1} = (4,-4)^t$, $\mathbf{c^2} = (1,12)^t$ and $\tilde{t}$ a random exponential variable of parameter $l = 2$, so that its distribution function is $F_t(h) = 1 - \exp(-2h)$ and its expected value and standard deviation are $\bar{t} = 0.5$ y $u = 0.5$, respectively, and consequently the expected value and the standard deviation of the random variable $\tilde{\mathbf{c}}^{\mathbf{t}}\mathbf{x}$ are $\bar{\mathbf{c}}^{\mathbf{t}}\mathbf{x} = 4.5x_1 + 2x_2$ and $s(\mathbf{x}) = 0.5x_1 + 6x_2$, respectively.

Let us proceed to solve the above problem using the solution criteria seen in the study and illustrate the results obtained.

The expected value and minimum variance problems associated with the stochastic problem have a unique solution. These solutions are $\mathbf{x_E}=(0, 1)^t$ and $\mathbf{x_V}=(2/3, 0)^t$, respectively, as indicated in Figure 1.

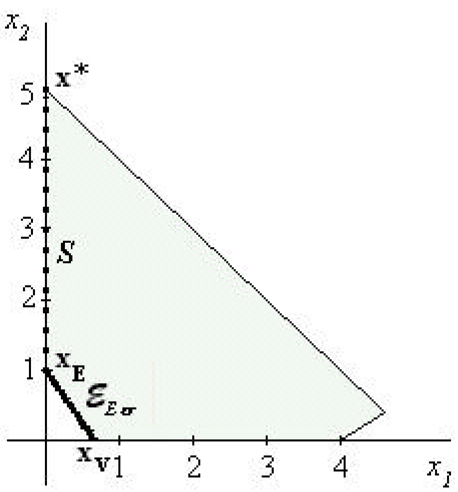

Figure 1

The set of expected value standard deviation efficient solutions, E_{Es}, can be obtained by solving the bi–criteria problem:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Min}}\ \left(4.5x_1 + 2x_2, 0.5x_1 + 6x_2\right) \\ &\text{s.t}\quad 3x_1 + 2x_2 \geq 2 \\ &\qquad 2x_1 - 3x_2 \leq 8 \\ &\qquad x_1 + x_2 \leq 5 \\ &\qquad x_1, x_2 \geq 0 \end{aligned} \tag{4.2}$$

and therefore the pay-off matrix for this problem is shown in Table 1.

Criterion	Solution	Expected Value	Standard Deviation
Expected value	$x_E=(0, 1)^t$	2	6
Variance	$x_V=(2/3, 0)^t$	3	1/3

Table 1: Pay-Off Matrix

The set of expected value standard deviation efficient solutions is:

$$E_{Es} = \left\{(x_1,\ x_2)^t \in R^2 / 3x_1 + 2x_2 = 2, x_1,\ x_2 \geq 0\right\}$$

This set is the segment of the feasible set which joins the vertices $\mathbf{x_E}$ and $\mathbf{x_V}$ in Figure 1. Note that these two solutions (which are unique optima for the expected value and minimum variance problems respectively) are expected value standard deviation efficient solutions. In addition, given that (4.2) is a linear problem, it is held as true that the sets of efficient and expected value standard deviation properly efficient solutions coincide, that is: $E_{Es} = E^p_{Es}$.

Let us now consider the solution of the problem using Kataoka criterion. For this, the feasible points of the problem must verify $\mathbf{c}^{\mathbf{2t}}\mathbf{x} = x_1 + 12x_2 > 0$. It is easy to prove that in this problem this condition holds true. The solutions to the deterministic equivalent problem which correspond to the application of the

Kataoka criterion with probability b, are those expressed in the following linear problem:

$$\begin{aligned} \underset{\mathbf{x}}{\text{Min}}\ & 4x_1 - 4x_2 + F_t^{-1}(b)(x_1 + 12x_2) \\ \text{s.t}\ \ & 3x_1 + 2x_2 \geq 2 \\ & 2x_1 - 3x_2 \leq 8 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{4.3}$$

where $F_t^{-1}(b) = -0.5\ln(1-b)$.

Before showing the solutions to this problem for different probabilities, it is worth noting that, from the results obtained in Section 3, we know that there exists a reciprocity between the optimal solutions of this problem and the minimum risk optimal solutions, that is, if we fix a probability, b, and the problem (4.3) is solved, we can affirm that the optimal solution obtained is also the minimum risk optimal solution to the stochastic problem for an aspiration level equal to the objective function in the minimum of the problem (4.3).

Furthermore, the results obtained show that the optimal solution to (4.3) may belong to the set of expected value standard deviation efficient solutions, and specifically we can state that:

(a) If $b = F_t(\bar{t}) \approx 0.6321$, problem (4.3) coincides with the expected value problem whose optimal solution is $\mathbf{x_E}$.

(b) The set of optimal solutions to the class of problems (4.3), when β varies in the set verifying that $b > F_t(\bar{t}) = F_t(0.5) = 1 - 1/e \approx 0.6321$, coincides with the set of expected value standard deviation efficient solutions to the stochastic problem, given that in this problem $E_{Es} = E_{Es}^{p}$.

(c) For values of $b < F_t(\bar{t}) \approx 0.6321$, the optimal solution to problem (4.3) does not need to be expected value standard deviation efficient.

Now, Table 2 shows the optimal Kataoka solutions for different values of b. Moreover we evaluate the expected value and the standard deviation of the stochastic objective in each one of them.

The level u minimum, which appears in the third column, is $u = 4x_1 - 4x_2 - 0.5(\ln(1-b))(x_1 + 12x_2)$ and the set S, which is indicated in Fig. 1, is the segment of the feasible set which joins the vertices **x*** = (0, 5)^t and $\mathbf{x_E}$:

$$S = \left\{(x_1, x_2) \in R^2 / \ x_1 = 0, \ x_2 \in [1, 5]\right\}.$$

Probability	Solution	Level u minimum	Expected Value	Standard Dev.
$b \in$ (0, 0.4865828)	$\mathbf{x}$ = (0, 5)^t	u = −20−30ln(1−β)	10	300
$b =$ 0.4865828	$\mathbf{x} \in S$	u = 0	$4.5x_1 + 2x_2$	$0.5x_1 + 6x_2$
$b \in$ (0.4865828, 0.69163483)	$\mathbf{x_E}$=(0, 1)^t	u = −4−6ln(1−β)	2	6
$b =$ 0.69163483	$\mathbf{x} \in E_{E\sigma}$	u = 3.058823	$4.5x_1 + 2x_2$	$0.5x_1 + 6x_2$
$b \in$ (0.69163483, 1)	$\mathbf{x_V}$=(2/3, 0)^t	u=1/3(8−ln(1−β))	3	1/3

Table 2: Optimal Kataoka Solutions

As is shown in Table 2, the optimal solutions to the problem (4.3) vary in accordance with the fixed probability and are determined by the vertices **x***, $\mathbf{x_E}$ and $\mathbf{x_V}$. From the results in Table 2 we can make the following comments:

If we look at rows 2, 4 and 6 in Table 2, we can note that the optimal solution to the Kataoka problem does not vary when a probability is set which belongs to any one of the three intervals. This tells us that in terms of choice of optimal solution there is no difference in setting, for example, the probabilities b = 0.1 or b = 0.4, since in the two cases the optimal choice is x_1= 0, x_2= 5 and, in addition, the expected value and the standard deviation is the same, being 10 and

300 respectively. In contrast, the level u minimum which reaches the objective function in the optimum of the problem (4.3) does vary according to the probability. In particular, it increases while the probability grows, which is logical since a higher probability and therefore less risky one, implies a solution of a worse level (remember that what is aimed for is the minimisation of the stochastic objective function). Thus, if the probability fixed in the Kataoka problem is $b \in (0, 0.4865828)$ the optimal solution for it is $\mathbf{x}^* = (0, 5)^t$ and, consequently, we can affirm that with probability b the stochastic objective function does not surpass level u, where $u = -20 - 30\ln(1-b)$. Note, too, that this solution, $\mathbf{x}^* = (0, 5)^t$, does not belong to the expected value standard deviation efficient set, as it corresponds to the fixing of a "low" probability in the Kataoka problem, and, in these cases, the Kataoka solution does not need to be an expected value standard deviation efficient solution.

If a probability b is fixed between 0.4865828 and 0.69163483 the Kataoka solution is the expected value solution, $\mathbf{x_E}=(0, 1)^t$ and the level u minimum increases with the probability, where $u = -4-6\ln(1-b)$. Finally, if the probability fixed is greater than 0.69163483, the Kataoka optimal solution is the minimum variance solution and, in this case, the level u minimum is $u = (8 - \ln(1-b))/3$, increasing with the probability fixed.

On the other hand, there exist two values of b in which the Kataoka problem has infinite solutions. For $b = 0.69163483$ the Kataoka problem has an infinite number of solutions which coincide, precisely, with the expected value standard deviation efficient set and, therefore, these solutions are all "equal" if we take note of the Kataoka optimum or minimum risk optimum criteria, (for the reciprocity existing between these two criteria), but they are not comparable if we focus on the expected value standard deviation efficiency criterion. However, where b = 0.4865828 the optimal solutions to the problem are the points belonging to segment S. Observe how, once again, these solutions are equal if we take note of the Kataoka criterion or the criterion of minimum risk, but, the expected value and the standard deviation of the aforesaid solutions varies from one solution to another. In particular, the expected value ranges between 10 and 2 and the standard deviation ranges between 30 and 6, such that if we observe the expected value standard deviation efficiency criterion, out of all of them we would choose the solution $\mathbf{x_E}$.

From all that has been stated above, we can affirm that in order to solve a stochastic programming problem we need to take into account different solution criteria for the problem, given that solutions which are equivalent in accordance with one of these criteria could turn out not to be so if we consider another.

5. Conclusions

The existing studies in the literature relating to the solving of stochastic programming problems reveal that, unlike in deterministic problems, the solving of stochastic problems, is by its very nature, more complicated, as it requires the choosing of a solution criterion. In this sense, it can be stated that these problems do not merely require the application of a more or less complicated problem solving approach.

The diversity of criteria involved in transforming the stochastic objective into its deterministic equivalent gives rise to different solution concepts apparently unrelated, since each solution criterion draws on a statistical characteristic of the stochastic objective function. The study carried out in this paper shows that these solution concepts are, under certain conditions, closely related and this may be of use to us when solving stochastic programming problems which prove as true these conditions. Thus, the reciprocity, which exists between the solution concepts of minimum risk and Kataoka, when the distribution function of the stochastic objective is strictly increasing, makes it possible to solve the stochastic problem by means of one of these criteria. In conjunction with this reciprocity is the relation established when certain hypotheses are true, between optimal solutions of any of these problems and the set of expected value standard deviation efficient solutions to the stochastic problem.

These relations reveal, moreover, "the wealth" of the Kataoka and minimum risk criteria for the solution of stochastic programming problems which prove as true the hypotheses established, given that it makes it possible to obtain a wide range of solutions which include, amongst others, the expected value standard deviation properly efficient solutions.

We need to point out, finally, that the choice of any of these criteria may not be sufficient for the solving of the stochastic problem, that is, in general it is not enough to choose the solution criterion which is considered best based on the preferences of the DM, solve the problem and show the solution(s) to the DM, as it is set out in the example, given that solutions which are "equal" in accordance with one of these criteria may not be so if they are considered in the light of another criterion, owing to the stochastic nature of the problem. In fact, in some cases, the solution by one criterion can produce a new decision-making process, which must be carried out in accordance with the preferences of the DM, in the way shown in the example.

References

Bereanu B. (1964). Programme de Risque Minimal en Programmation Linéaire Stochastique. *C. R. Acad. Sci. Paris* 259, 981-983.

Charnes A. and Cooper W.W. (1963). Deterministic Equivalents for Optimizing and Satisfying under Chance Constraints. *Operations Research* 11, 1, 18-39.

Kall P. (1982). Stochastic Programming. *European Journal of Operational Research* 10, 125-130.

Kall P. and Wallace S.W. (1994). *Stochastic Programming*. John Wiley and Sons. Chichester.

Kataoka S. (1963). A Stochastic Programming Model. *Econometrica* 31, 181-196.

Kibzun A.I. and Kan I.S. (1996). *Stochastic Programming Problems with Probability and Quantile Functions*. John Wiley and Sons. Chichester.

Leclerq J.P. (1982). Stochastic Programming: An Interactive Multicriteria Approach. *European Journal of Operational Research* 10 , 33-41.

Markowitz H. (1952). Portfolio Selection. *The Journal of Finance* 7, 77-91.

Prékopa A. (1995). *Stochastic Programming*. Kluwer Academic Publishers. Dordrecht.

Sawaragi Y., Nakayama H. and Tanino T. (1985). *Theory of Multiobjective Optimization*. Academic Press. New York.

Stancu-Minasian I.M. (1984). *Stochastic Programming with Multiple Objective Functions*. D. Reidel Publishing Company. Dordrecht.

Zare Y. and Daneshmand A. (1995). A Linear Approximation method for solving a special class of the chance constrained programming problem. *European Journal of Operational Research* 80, 213-225.